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Imprints of log-periodic self-similarity in the stock market

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Abstract. Detailed analysis of the log-periodic structures as precursors of the financial crashes is presented. The study is mainly based on the German Stock Index (DAX) variation over the 1998 period which includes both, a spectacular boom and a large decline, in magnitude only comparable to the so-called Black Monday of October 1987. The present example provides further arguments in favour of a discrete scale-invariance governing the dynamics of the stock market. A related clear log-periodic structure prior to the crash and consistent with its onset extends over the period of a few months. Furthermore, on smaller time-scales the data seems to indicate the appearance of analogous log-periodic oscillations as precursors of the smaller, intermediate decreases. Even the frequencies of such oscillations are similar on various levels of resolution. The related value $\lambda \approx 2$ of preferred scaling ratios is amazingly consistent with those found for a wide variety of other complex systems. Similar analysis of the major American indices between September 1998 and February 1999 also provides some evidence supporting this concept but, at the same time, illustrates a possible splitting of the dynamics that a large market may experience.

PACS. 01.75.+m Science and society – 05.40.-a Fluctuation phenomena, random processes, and Brownian motion – 89.90.+n Other topics of general interest to physicists

The fact that a healthy and normally functioning financial market may reveal certain properties common to complex systems is fascinating and, in fact, seems natural. Especially interesting in this context is the recently suggested analogy of the financial crashes to critical points in statistical mechanics [1–6]. Criticality implies a scale invariance which in mathematical terms, for a properly defined function $F(x)$ characterizing the system, means that for small x

$$F(\lambda x) = \gamma F(x). \tag{1}$$

A positive constant γ in this equation describes how the properties of the system change when it is rescaled by the factor λ. The simplest solution to this equation reads:

$$F_0(x) = x^\alpha, \tag{2}$$

where $\alpha = \log(\gamma)/\log(\lambda)$. This is a standard power-law that is characteristic of continuous scale-invariance and α is the corresponding critical exponent.

More interesting is the general solution [7] to equation (1):

$$F(x) = F_0(x) P(\log F_0(x)/\log(\gamma)), \tag{3}$$

where P denotes a periodic function of period one. In this way the dominating scaling (2) acquires a correction which is periodic in $\log(x)$. This solution accounts for a possible discrete scale-invariance [8] and can be interpreted [9,10] in terms of a complex critical exponent $\alpha = \alpha_R + i\alpha_I$, since $\Re\{x^\alpha\} = x^{\alpha_R}\cos(\alpha_I \log(x))$, which corresponds to the first term in a Fourier expansion of (3). Thus, if x represents a distance to the critical point, the resulting spacings between consecutive minima x_n (maxima) of the log-periodic oscillations seen in the linear scale follow a geometric contraction according to the relation:

$$\frac{x_{n+1} - x_n}{x_{n+2} - x_{n+1}} = \lambda. \tag{4}$$

Then, the critical point coincides with the accumulation of such oscillations.

Existence of the log-periodic modulations correcting the structureless pure power-law behaviour has been identified in many different systems [8]. Examples include diffusion-limited-aggregation clusters [11], crack growth [12], earthquakes [9,10] and, as already mentioned, the financial market where x is to be interpreted as the time to crash. Especially in the last two cases this is an extremely interesting feature because it potentially offers a tool for predictions. Of course, the real financial market is exposed to many external factors which may distort its internal hierarchical structure on the organizational as well as on the dynamical level. Therefore, the searches for the long term, of the order of few years, precursors of crashes

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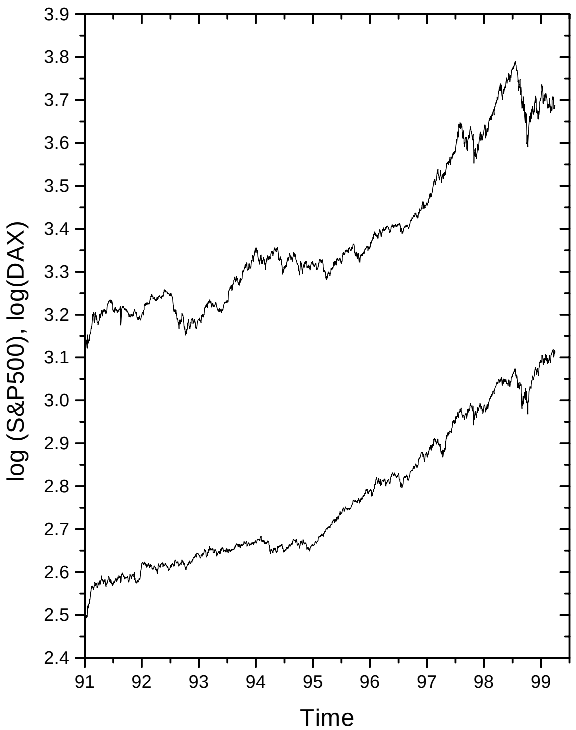

Fig. 1. The Deutsche Aktienindex–DAX (upper chart) *versus* S&P 500 (lower chart) in the period 1991-1999. Logarithms of both indices are shown for a better comparison within the same scale.

have to be taken with some reserve, as already pointed out in reference [13]. A somewhat related example is shown in Figure 1 which displays the S&P 500 *versus* DAX charts between 1991 and February 1999. While the global characteristics of the two charts are largely compatible there exist several significant differences on shorter time-scales. It is the purpose of the present paper to explore more in detail the emerging short-time behaviour of the stock market indices.

On the more general ground, the current attitude in developing the related theory is logically not fully consistent. First of all, no methodology is provided as how to incorporate a pattern of log-periodic oscillations preceding a particular crash into an even more serious crash which potentially may occur in a year or two later. Secondly, even though there is some indication that the largest crashes are outliers by belonging to a different population [14], there exists no precise definition of what is to be qualified as a crash, especially in the context of its analogy to critical points of statistical mechanics. Just a bare statement that the crash corresponds to discontinuity in the derivative of an appropriate market index is not sufficiently accurate to decide what amount of decline is needed to signal a real crash and what is to be considered only a "correction". In fact, a closer inspection of various market indices on different time-scales suggests that it is justifiable to consider them as nowhere differentiable. An emerging scenario of the market evolution, in a natural way resolving this kind of difficulties, would then correspond to a permanent competition between booms and crashes of various sizes; a picture somewhat analogous to the self-organized critical state [15] and consistent with a causal information cascade from large scales to small scales as demonstrated through the analysis of correlation functions [16]. In this connection the required existence of many critical points within the renormalization group theory may result from a more general nonlinear renormalization flow map, *i.e.* by replacing λx by $\phi(x)$ [17,18]. In fact, such a mechanism may even remain compatible with the log-periodic scaling properties of equation (3) on various scales. For this to apply the accumulation points of the log-periodic oscillations on smaller scales need themselves to be distributed as the log-periodic sequence.

Identification of a clean hierarchy of the above suggested structures on the real market is not expected to be an easy task because of a possible contamination by various external factors or by some internal market nonuniformities. However, on the longer time-scales many such factors may cancel out to a large extent. Within the shorter time-intervals, on the other hand, the influence of such factors can significantly be reduced by an appropriate selection of the location of such intervals. In this later sense one finds the most preferential conditions in the recent DAX behaviour as no obvious external events that may have influenced its evolution can be indicated. Here, as illustrated in Figure 2, within the period of only 9 month preceding July 1998 the index went up from about 3700 to almost 6200 and then quickly declined to below 4000. This draw down is however somewhat slower than some of the previous crashes analysed in similar context but for this reason it even better resembles a real physical second order phase transition. During the spectacular boom period the three most pronounced deep minima, indicated by the upward long solid-line arrows, can immediately be located. Denoting the resulting times as t_n, t_{n+1}, t_{n+2} and making the correspondence with equation (3) by setting $x_{\rm i} = t_{\rm c} - t_{\rm i}$, where $t_{\rm c}$ is the crash time, already such three points can be used to determine $t_{\rm c}$:

$$t_{\rm c} = \frac{t_{n+1}^2 - t_{n+2}t_n}{2t_{n+1} - t_n - t_{n+2}} \cdot \qquad (5)$$

The result is indicated by a similar downward arrow and reasonably well agrees with the actual time of crash. The corresponding preferred scaling ratio between $t_{n+1} - t_n$ and $t_{n+2} - t_{n+1}$ (Eq. (4)), governing the log-periodic oscillations, gives $\lambda = 2.17$, which is consistent with the previous cases analysed in the literature not only in connection with the market evolution [19] but for a wide variety of other systems as well [8] and, thus, may indicate a universal character of the mechanism responsible for discrete scale invariance in complex systems.

As a further element of the present analysis it can be quite clearly seen from Figure 2 that there is essentially no qualitative difference between the nature of the major crash and those index declines that mark its preceding log-periodically distributed minima. Indeed, they also seem to be preceded by their own log-periodic oscillations within appropriately shorter time-intervals. The two such sub-sequences are indicated by the long-dashed and

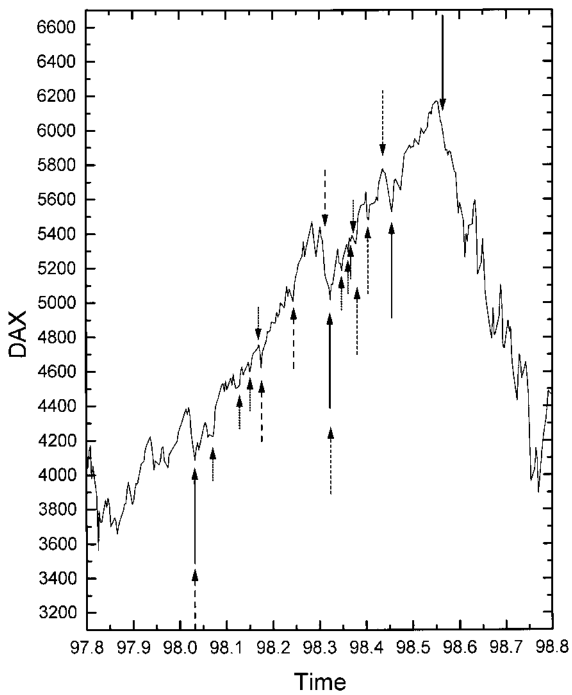

Fig. 2. The daily evolution of the Deutsche Aktienindex from October 1997 to October 1998. Upward arrows indicate minima of the log-periodic oscillations used to determine the corresponding critical times denoted by the downward arrows. Different types of arrows (three upward and one downward) correspond to different sequences of log-periodic oscillations identified on various time-scales.

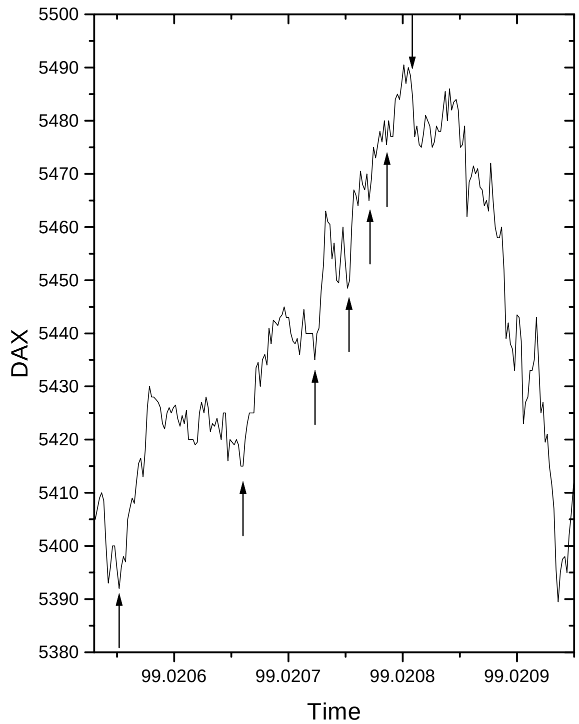

Fig. 3. The minutely DAX variation between 11:50 and 15:30 on January 8, 1999. Upward arrows indicate minima used to determine the corresponding critical time.

short-dashed arrows, respectively, and the corresponding t_c calculated using equation (5) by downward arrows of the same type. In both cases the so-estimated t_c's also reasonably well coincide with times of the decline. Interestingly, the scaling ratios λ for these log-periodic structures equal 2.06 and 2.07, respectively, and thus turn out consistent with the above value of 2.17. Moreover, even on the deeper level of resolution the two sequences of identifiable oscillations indicated by the dotted-line arrows in Figure 2 develop analogous structures resulting in $\lambda = 2.26$ (earlier case) and $\lambda = 2.1$ (later case), which is again consistent with all its previous values. This means that such a whole fractal hierarchy of log-periodic structures may still remain grasped by one function of the type (3). In this scenario the largest crash of Figure 2 may appear as just one component of log-periodic oscillations extending into the future and announcing an even larger future crash. Large crashes can thus be assigned no particular role in this sense. They are preceded by the log-periodic oscillations of about the same frequency as the small ones. What still can make them outliers [14] are parallel secondary effects like an overall increase of the market volume which may lead to an additional amplification of their amplitude.

Violent reverse in a market tendency may reveal log-periodic-like structures even during the intra-day trading. One such example is illustrated in Figure 3 which shows the minutely DAX variation between 11:50 and 15:30 on January 8, 1999. It is precisely during this period (still seen in Fig. 1) that DAX reached its few month maximum after recovery from the previously discussed crash. Taking the average of the ratios between the consecutive five neighbouring time-intervals determined by the six deepest minima indicated by the upward arrows results here in $\lambda \approx 1.7$. The corresponding t_c (downward arrow) again quite precisely indicates the onset of the decline.

Directly before the major crash on July 20, 1998 the trading dynamics somewhat slows down (as can be seen from Fig. 2) and no such structures on the level of minutely variation can be identified in this case, however. In fact, a fast increase of the market index just before its subsequent decline seems to offer the most favourable conditions for the log-periodic oscillations to show up on the time-scales of a few month or shorter. This may simply reflect the fact that a faster internal market dynamics generates such oscillations of larger amplitude which thus gives them a better chance to dominate a possible external corruption. Consistently, another market (Hong-Kong Stock Exchange) whose Hang Seng index went up recently by almost 40% during about a four month period in 1997 also provides quite a convincing example of the short term log-periodic oscillations. The four most pronounced minima at 97.24, 97.43, 97.52 and 97.56 trace a geometric progression with a common λ of about 2.15 and this progression converges to $t_c \approx 97.6$, thus exactly indicating the beginning of a dramatic crash. The relevant Hang Seng chart can be seen in Figure 2 of reference [6].

Instead of listing further examples where the log-periodic oscillations accompany a local fast increase of the market index we find it more instructive to study more

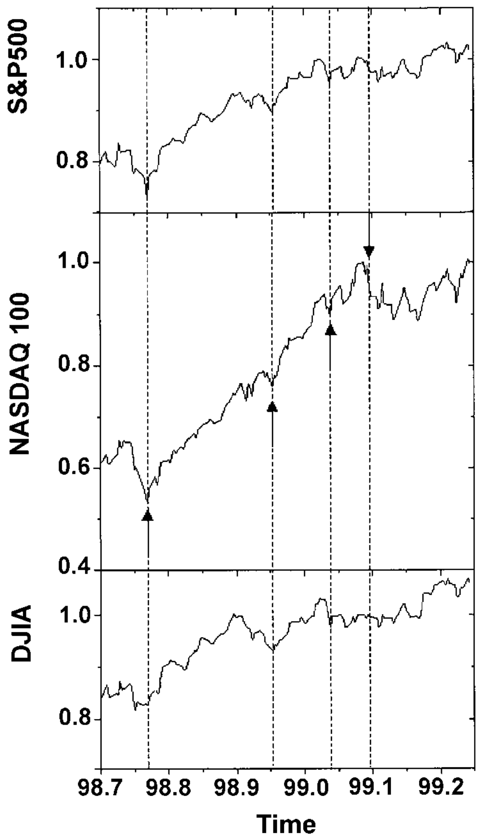

Fig. 4. The daily variation of S&P500 (upper panel), Nasdaq-100 (middle panel) and Dow Jones Industrial Average (lower panel) from September 1998 till March 1999. All these indices are normalized such they equal unity on February 1, 1999.

in detail the recent development on the American market. It provides further evidence in favour of this concept but at the same time illustrates certain possible related subtleties. Figure 4 shows the S&P500 behaviour starting mid September 1998 *versus* the two closely related and most frequently quoted indices: the Dow Jones Industrial Average (DJIA) which is entirely comprised by S&P500 and the Nasdaq-100 whose about 60% of the volume overlaps with S&P500. The latter two indices (DJIA and Nasdaq-100) are totally disconnected in terms of the company content. In order to make the relative speed of the changes directly visible all these indices are normalized such that they are equal to unity at the same date (here on February 1, 1999). Clearly, it is Nasdaq which within the short period between October 8, 1998 (its lowest value in the period considered) and February 1, 1999 develops a very spectacular rise by almost doubling its magnitude. In this case the three most pronounced consecutive minima ($\lambda = 2.25$) also quite precisely point to the onset of the following 11% correction. Parallel increase of the DJIA is much slower, the pattern of oscillations much more difficult to uniquely interpret and, consistently, no correction occurs. At the same time the S&P500 largely behaves like an average of the two. Even though it displays similar three minima as Nasdaq, the early February correction is only rudimentary. In a sense we are thus facing an example of a very interesting temporary spontaneous decoupling of a large market, as here represented by S&P500, into submarkets some of which may evolve for a certain period of time according to their own log-periodic pattern of oscillations which are masked in the global index. The smaller, more uniform markets are less likely to experience such effects of decoupling and are thus expected to constitute better candidates to manifest the short-time universal structures. The opposite may apply to the global index since from the longer time-scales perspective such effects should be less significant. The examples analysed here as well as those studied in the literature are in fact consistent with this interpretation. If applies, such an interpretation provides another physically appealing picture: short-time log-periodicity is more localized in the "market space" while the longer-time-scales probe its more global aspects.

Tabulating the critical exponents α_R for the stock market in the context of our present study of criticality on various time-scales doesn't seem equally useful as their values significantly depend on the time-window inspected. For instance, in the case of the DAX index we find values ranging between 0.2 for the few years-long time-intervals up to almost 1 for the shorter time-intervals corresponding to the identified log-periodic substructures. One may argue that the logarithm of the stock market index constitutes a more appropriate quantity for determining the critical exponents. However, also on the level of the logarithm the exponents vary between about 0.3 up to 1 in analogous time-intervals as above.

In conclusion, the present analysis provides further arguments for the existence of the log-periodic oscillations constituting a significant component in the time-evolution of the fluctuating part of the stock market indices. Even more, imprints are found for the whole hierarchy of such log-periodically oscillating structures on various time-scales and this hierarchy carries signatures of self-similarity. An emerging scenario of the market evolution characterized by nowhere differentiable permanent competition between booms and crashes of various size is then much more logically acceptable and consistent. Of course, in general, it would be naive to expect that on the real market any index fluctuation can uniquely be classified as a member of a certain log-periodically distributed sequence. Some of such fluctuations may be caused by external factors which are likely to be completely random relative to the market intrinsic evolutionary synchrony. It is this complex intrinsic interaction of the market constituents which may lead to such universal features as the ones discussed above and it is extremely interesting to see that such features (a consistent sequence of the log-periodically distributed oscillations) can quite easily be identified with help of some physics guidance. The above result makes also clear that the stock market log-periodicity reveals much richer structure than just lowest order Fourier expansion of equation (3) [1–3], and therefore, at the present stage the "arrow dropping" procedure used here offers much more flexibility in catching the essential structures and seems thus more appropriate, especially on shorter time-scales. Finally, in this context we wish to draw attention to the Weierstrass-Mandelbrot fractal function[1] which is continuous everywhere, but is nowhere differentiable [8,20] and can be made to obey the renormalization group equation. The relevance of this function for log-periodicity has already been pointed out in connection with earthquakes [21]. It is likely that a variant of this function also provides an appropriate representation for the stock market criticality.

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[1] The Weierstrass-Mandelbrot fractal function is defined as $W(t) = \sum_{n=-\infty}^{\infty} (1 - \mathrm{e}^{\mathrm{i}\gamma^n t})\mathrm{e}^{\mathrm{i}\phi_n}/\gamma^{\eta n}$, where $0 < \eta < 1$, $\gamma < 1$ and ϕ_n is an arbitrary phase. It is easy to show by relabeling the series index that for an appropriate choice of the set of phases $\{\phi_n\}$, $W(\gamma t) = \gamma^\eta W(t)$.